WIENER'S LEMMA FOR INFINITE MATRICES II

QIYU SUN

Abstract. In this paper, we introduce a class of infinite matrices related to the Beurling algebra of periodic functions, and we show that it is an inverse-closed subalgebra of $\mathcal{B}(\ell^q_w)$, the algebra of all bounded linear operators on the weighted sequence space ℓ^q_w, for any $1 \le q < \infty$ and any discrete Muckenhoupt A_q-weight w.

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1. Introduction

Let us begin this sequel to [44] by introducing a new class of infinite matrices,

$$\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d) := \Big\{A := (a(i,j))_{i,j\in\mathbb{Z}^d} \;\Big|\; \|A\|_{\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)} < \infty\Big\}, \tag{1.1}$$

where $d \ge 1$, $|x|_\infty := \max(|x_1|,\ldots,|x_d|)$ for $x := (x_1,\ldots,x_d) \in \mathbb{R}^d$, and

$$\|A\|_{\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)} := \sum_{k\in\mathbb{Z}^d}\Big(\sup_{|i-j|_\infty\ge|k|_\infty}|a(i,j)|\Big).$$

It is observed that a Laurent matrix $A := (a(i-j))_{i,j\in\mathbb{Z}}$ associated with a sequence $a := (a(n))_{n\in\mathbb{Z}}$ belongs to $\mathcal{B}(\mathbb{Z},\mathbb{Z})$ if and only if the Fourier series $\hat a(\xi) := \sum_{n\in\mathbb{Z}} a(n)\exp(-\sqrt{-1}\, n\xi)$ belongs to the Beurling algebra

$$A^*(\mathbb{T}) := \Big\{\sum_{n=-\infty}^{\infty} a(n)e^{-\sqrt{-1}\, n\xi} \;\Big|\; \sum_{k=0}^{\infty}\sup_{|n|\ge k}|a(n)| < \infty\Big\}.$$

The algebra $A^*(\mathbb{T})$ was introduced by A. Beurling for establishing contraction properties of periodic functions [9], and was used in considering pointwise summability of Fourier series [11, 16, 17, 41, 49]. So the class $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$ of infinite matrices can be interpreted as a noncommutative matrix extension of the *Beurling algebra* $A^*(\mathbb{T})$.

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Define the *Gröchenig-Schur class* $\mathcal{S}(\mathbb{Z}^d,\mathbb{Z}^d)$ by

$$\begin{aligned}\mathcal{S}(\mathbb{Z}^d,\mathbb{Z}^d) \ := \ & \Big\{\big(a(i,j)\big)_{i,j\in\mathbb{Z}^d} \ \Big| \\ & \max\Big(\sup_{i\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}|a(i,j)|, \sup_{j\in\mathbb{Z}^d}\sum_{i\in\mathbb{Z}^d}|a(i,j)|\Big)<\infty\Big\} \end{aligned} \tag{1.2}$$

[26, 38, 42, 44], and the *Gohberg-Baskakov-Sjöstrand class* $\mathcal{C}(\mathbb{Z}^d,\mathbb{Z}^d)$ by

$$\mathcal{C}(\mathbb{Z}^d,\mathbb{Z}^d) := \Big\{\big(a(i,j)\big)_{i,j\in\mathbb{Z}^d} \ \Big| \ \sum_{k\in\mathbb{Z}^d}\big(\sup_{i-j=k}|a(i,j)|\big)<\infty\Big\} \tag{1.3}$$

[6, 21, 26, 33, 40, 44]. The above two classes of infinite matrices appeared in the study of Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators etc (see [2, 4, 25, 29, 40, 43] for a sample of papers). From (1.1), (1.2) and (1.3) it follows that

$$\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)\subset\mathcal{C}(\mathbb{Z}^d,\mathbb{Z}^d)\subset\mathcal{S}(\mathbb{Z}^d,\mathbb{Z}^d).$$

Hence any matrix in $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$ belongs to the Gröchenig-Schur class $\mathcal{S}(\mathbb{Z}^d,\mathbb{Z}^d)$ and also the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d,\mathbb{Z}^d)$.

An equivalent way of defining an element $A := (a(i,j))_{i,j\in\mathbb{Z}^d}$ in $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$ is the existence of a radially decreasing sequence $\{b(i)\}_{i\in\mathbb{Z}^d}$ such that

$$|a(i,j)|\le b(i-j) \quad \text{for all } i,j\in\mathbb{Z}^d,$$

$$\sum_{i\in\mathbb{Z}^d} b(i)<\infty,$$

and

$$b(i)=h(|i|_\infty) \text{ for some decreasing sequence } \{h(n)\}_{n=0}^\infty.$$

Therefore any infinite matrix in $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$ is dominated by a convolution operator associated with a summable, radial and (radially) decreasing sequence. We remark that any infinite matrix in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d,\mathbb{Z}^d)$ is dominated by a convolution operator associated with a summable sequence [6, 21, 26, 33, 40, 44].

A positive sequence $w := (w(i))_{i\in\mathbb{Z}^d}$ is said to be a *discrete A_q-weight* for $1<q<\infty$ if there exists a positive constant C such that

$$\Big(N^{-d}\sum_{i\in a+[0,N-1]^d} w(i)\Big)\Big(N^{-d}\sum_{i\in a+[0,N-1]^d}(w(i))^{-1/(q-1)}\Big)^{q-1}\le C \tag{1.4}$$

hold for all $a \in \mathbb{Z}^d$ and $1 \le N \in \mathbb{Z}$, and to be a *discrete* A_1*-weight* if there exists a positive constant C such that

$$N^{-d} \sum_{i \in a+[0,N-1]^d} w(i) \le C \inf_{i \in a+[0,N-1]^d} w(i) \tag{1.5}$$

hold for all $a \in \mathbb{Z}^d$ and $1 \le N \in \mathbb{Z}$ [19, 41]. The *discrete* A_q*-bound*, denoted by $A_q(w)$, is the smallest constant C for which (1.4) holds when $1 < q < \infty$, and respectively for which (1.5) holds when $q = 1$. The positive sequences $((1+|i|_\infty)^\alpha)_{i\in\mathbb{Z}^d}$ with $-d < \alpha < d(q-1)$ if $1 < q < \infty$, and respectively with $-d < \alpha \le 0$ if $q = 1$, are discrete A_q-weights.

For $1 < q < \infty$, a positive locally-integrable function w on $\mathbb{R}^d$ is said to be an A_q*-weight* if there exists a positive constant C such that

$$\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-1/(q-1)}dx\Big)^{q-1} \le C \tag{1.6}$$

for all cubes $Q \subset \mathbb{R}^d$, where $|Q|$ denotes the Lebesgue measure of the cubic Q [37]. Similarly for $q = 1$, a positive locally-integrable function w is said to be an A_1*-weight* if there exists a positive constant C such that

$$\frac{1}{|Q|}\int_Q w(y)dy \le Cw(x), \qquad x \in Q \tag{1.7}$$

for all cubes $Q \subset \mathbb{R}^d$ [15]. One may then verify that for $1 \le q < \infty$, a positive sequence $w := \big(w(i)\big)_{i\in\mathbb{Z}^d}$ is a discrete A_q-weight if and only if $\tilde{w}(x) := \sum_{i\in\mathbb{Z}^d} w(i)\chi_{[-1/2,1/2)^d}(x-i)$ is an A_q-weight, where χ_E is the characteristic function on a set E [34].

For $1 \le q < \infty$ and a positive sequence $w := (w(i))_{i\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$, let $\ell^q_w := \ell^q_w(\mathbb{Z}^d)$ be the space of all weighted q-summable sequences on $\mathbb{Z}^d$, i.e.,

$$\ell^q_w(\mathbb{Z}^d) := \Big\{(c(i))_{i\in\mathbb{Z}^d} \;\Big|\; \|c\|_{q,w} := \Big(\sum_{i\in\mathbb{Z}^d} |c(i)|^q w(i)\Big)^{1/q} < \infty\Big\}.$$

For the trivial weight w_0 (i.e. $w_0(i) = 1$ for all $i \in \mathbb{Z}^d$), we will use ℓ^q and $\|\cdot\|_q$ instead of $\ell^q_{w_0}$ and $\|\cdot\|_{q,w_0}$ for brevity. Define the *discrete maximal function* by

$$Mc(i) := \sup_{0\le N\in\mathbb{Z}} \frac{1}{(2N+1)^d} \sum_{k\in i+[-N,N]^d} |c(k)| \quad \text{for } c := (c(i))_{i\in\mathbb{Z}^d}.$$

Similar to the characterization of an A_q-weight on $\mathbb{R}^d$ via the standard maximal operator [35], the discrete A_q-weight can be characterized via the discrete maximal function on the weighted ℓ^q space. More precisely, a positive sequence $w := (w(i))_{i\in\mathbb{Z}^d}$ is a discrete A_q-weight if and only if the discrete maximal operator $c \longmapsto Mc$ is of weak-type (ℓ^q_w, ℓ^q_w), i.e., there exists a positive constant C such that

$$\sum_{Mc(i)\ge\alpha} w(i) \le \frac{C}{\alpha^q}\|c\|^q_{q,w} \quad \text{for all } \alpha>0 \text{ and } c\in\ell^p_w.$$

Moreover for $1<q<\infty$, the discrete maximal operator M is of strong type (ℓ^q_w, ℓ^q_w) for a discrete A_q-weight w, i.e., there exists a positive constant C' such that

$$\|Mc\|_{q,w} \le C'\|c\|_{q,w} \quad \text{for all } c\in\ell^p_w.$$

The reader may refer to [19] for a complete account of the theory of weighted inequalities and its ramifications.

Now let us present our results for infinite matrices in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$. In Section 3, we establish the following algebraic properties for the class $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices.

Theorem 1.1. *Let $1\le q<\infty$, and let w be a discrete A_q-weight. Then $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is a unital Banach algebra under matrix multiplication, and it is also a subalgebra of $\mathcal{B}(\ell^q_w)$, the algebra of all bounded linear operators on the weighted sequence space ℓ^q_w.*

By Theorem 1.1, every infinite matrix $A\in\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ defines a bounded operator on ℓ^q_w for any $1\le q<\infty$ and for any discrete A_q-weight w, i.e., there exists a positive constant C such that

$$\|Ac\|_{q,w} \le C\|c\|_{q,w} \quad \text{for all } c\in\ell^q_w,$$

see (3.16) for an estimate of the above constant C. Besides the above boundedness of an infinite matrix on the weighted sequence space ℓ^q_w, it is natural to consider ℓ^q_w-stability. Here for $1\le q<\infty$ and a positive sequence w on $\mathbb{Z}^d$, we say that an infinite matrix A has *ℓ^q_w-stability* if there exists a positive constant C such that

$$C^{-1}\|c\|_{q,w} \le \|Ac\|_{q,w} \le C\|c\|_{q,w} \quad \text{for all } c\in\ell^q_w.$$

The ℓ^q_w-stability is one of the basic assumptions for infinite matrices arising in the study of spline approximation, Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators etc (see [2, 4, 25, 29, 30, 40, 43, 44] and the references therein.) In Section 4, we establish the equivalence of ℓ^q_w-stabilities of any infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ for different exponents $1\le q<\infty$ and for different discrete A_q-weights w.

Theorem 1.2. *Let $A \in \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$. If A has ℓ^q_w-stability for some $1 \le q < \infty$ and for some discrete A_q-weight w, then it has $\ell^{q'}_{w'}$-stability for all $1 \le q' < \infty$ and for all discrete $A_{q'}$-weights w'.*

The reader may refer to [1, 39, 50] for the equivalence of unweighted ℓ^q_w-stability of infinite matrices in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$. If $A \in \mathcal{B}(\ell^q_w)$ has a *left inverse* $B \in \mathcal{B}(\ell^q_w)$, i.e., $BA = I$, then A has ℓ^q_w-stability. The converse is not true in general, unless $q = 2$. As an application of Theorem 1.2, we show that the converse holds for any infinite matrix A in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$.

Corollary 1.3. *Let $1 \le q < \infty$, and let w be a discrete A_q-weight. Then an infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ has ℓ^q_w-stability if and only if it has a left inverse in $\mathcal{B}(\ell^q_w)$.*

Given a Banach algebra $\mathcal{B}$, a subalgebra $\mathcal{A}$ of $\mathcal{B}$ is said to be *inverse-closed* if $A \in \mathcal{A}$ and the inverse A^{-1} of the element A exists in $\mathcal{B}$ implies that $A^{-1} \in \mathcal{A}$ [20, 36, 48]. The next question following the ℓ^q_w-stability of an infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is whether its inverse, if exists in $\mathcal{B}(\ell^q_w)$, belongs to $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$, or in the other words, whether $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^q_w)$.

The inverse-closedness for the subalgebra of absolutely convergent Fourier series in the algebra of bounded periodic functions was first studied in [10, 20, 36, 51]. The inverse-closed property (=Wiener's lemma) has been established for infinite matrices satisfying various off-diagonal decay conditions, see [3, 5, 6, 18, 21, 24, 26, 30, 40, 42, 44] for a sample of papers. Inverse-closedness occurs under various names (such as spectral invariance, Wiener pair, local subalgebra) in many fields of mathematics, see the survey [22].

The inverse-closed property of non-commutative matrix subalgebras has been shown to be crucial for the well-localization of dual wavelet frames and dual Gabor frames [4, 25, 30, 31, 32], the algebra of pseudo-differential operators [23, 29, 40], the fast implementation in numerical analysis [13, 14, 28], and the local reconstruction in sampling theory [2, 43, 46].

In [24], Gröchenig and Klotz considered the forms of off-diagonal decay that can be inherited under matrix inversion and said that "*The answers so far mix art and hard mathematical work. The art is to guess a suitable decay condition, the work is then to prove that this decay condition is preserved under inversion.*" Particularly in our case, the art part is to find suitable algebras $\mathcal{A}$ of infinite matrices and the mathematical part is to prove the inverse-closedness of the algebra $\mathcal{A}$ in $\mathcal{B}(\ell^q_w)$. There are several approaches to prove the inverse-closedness of a subalgebra of $\mathcal{B}(\ell^q_w)$. Here are three of them: (i) the indirect approach, such as the Gelfand's technique [18, 20, 26]; (ii) the semi-direct approach, such as the bootstrap argument [30] and the derivation trick [24]; (iii) the direct approach, such as the commutator trick [6, 40], the decomposition technique based on the Bochner-Phillips theorem [6, 7, 21], and the asymptotic estimate technique [39, 42, 44]. Each approach has its advantages and limitations. For instance, the Gelfand technique and the asymptotic estimate technique work well for inverse-closed subalgebras of $\mathcal{B}(\ell^q)$ with $q=2$, but they are not directly applicable for inverse-closed subalgebras of $\mathcal{B}(\ell^q)$ with $q\neq 2$. The commutator trick is applicable to establish Wiener's lemma for subalgebra of $\mathcal{B}(\ell^q), 1\le q\le\infty$ [6, 40]. In Section 5, we combine the commutator trick, the asymptotic estimate technique and the equivalence of ℓ^q_w-stability for different exponents q and for different discrete A_q-weights w, and then establish Wiener's lemma for subalgebras of $\mathcal{B}(\ell^q_w)$, where $1\le q<\infty$ and w is a discrete A_q-weight.

Theorem 1.4. *Let $1\le q<\infty$ and let w be a discrete A_q-weight. Then $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^q_w)$.*

Remark 1.5. Let $1\le q<\infty$, w be a discrete A_q-weight, and $A=(a(i,j))_{i,j\in\mathbb{Z}^d}\in\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$ have a bounded inverse in $\mathcal{B}(\ell^q_w)$. Inspection of the proof of Theorem 1.4 shows that the norm $\|A^{-1}\|_{\mathcal{B}}$ of the inverse matrix A^{-1} depends on the exponent q, the weight w and the matrix A not only through the discrete A_q-bound $A_q(w)$, the norm $\|A\|_{\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)}$ in the algebra $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$, the operator norm $\|A^{-1}\|_{\mathcal{B}(\ell^q_w)}$ in the operator algebra $\mathcal{B}(\ell^q_w)$, and also through a rather **implicit** quantity N such that $A_q(w)^{2/q}\sum_{|\mathbf{k}|_\infty\ge\sqrt{N}/2}\sup_{|i-j|_\infty\ge|k|_\infty}|a(i,j)|$ is sufficiently small.

As an application of Theorem 1.4, we obtain Wiener's lemma for the Beurling algebra $A^*(\mathbb{T})$ of periodic functions [8].

Corollary 1.6. *If $f\in A^*(\mathbb{T})$ and $f(\xi)\neq 0$ for all $\xi\in\mathbb{R}$ then $1/f\in A^*(\mathbb{T})$.*

As applications of Theorems 1.2 and 1.4, we establish the equivalence between the ℓ^q_w-stability of an infinite matrix in $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$ and the existence of its left inverse in $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$.

Corollary 1.7. *Let $1\le q<\infty$, and let w be a discrete A_q-weight. Then an infinite matrix in $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$ has ℓ^q_w-stability if and only if it has a left inverse in $\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)$.*

2. A class of infinite matrices

In this section, we introduce a class of infinite matrices with off-diagonal decay, which includes the class $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ in the Introduction as a special case.

A *weight matrix on* $\mathbb{Z}^d \times \mathbb{Z}^d$, or a *weight matrix* for brevity, is a positive matrix $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$ with each entry not less than one, i.e., $u(i,j) \ge 1$ for all $i, j \in \mathbb{Z}^d$. For $1 \le p \le \infty$ and a weight matrix $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$, define

$$\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ A := (a(i,j))_{i,j\in\mathbb{Z}^d} \ \Big| \ \|A\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} < \infty \Big\}, \tag{2.1}$$

where

$$\|A\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} := \Big\| \Big(\sup_{|i-j|_\infty \ge |k|_\infty} |a(i,j)| u(i,j) \Big)_{k\in\mathbb{Z}^d} \Big\|_p. \tag{2.2}$$

If $p = 1$ and $u \equiv 1$ (i.e., all entries of the weight matrix u are equal to 1), then

$$\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) = \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) \quad \text{and} \quad \|\cdot\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} = \|\cdot\|_{\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)}.$$

In this paper, we use $\mathcal{B}_{p,u}, \mathcal{B}, \|\cdot\|_{\mathcal{B}_{p,u}}, \|\cdot\|_{\mathcal{B}}$ instead of $\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d), \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$, $\|\cdot\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)}, \|\cdot\|_{\mathcal{B}(\mathbb{Z}^d,\mathbb{Z}^d)}$ for brevity.

Remark 2.1. Let $1 \le p \le \infty$ and u be a weight matrix. Define the *Gröchenig-Schur class* $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices by

$$\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ A := (a(i,j))_{i,j\in\mathbb{Z}^d} \ \Big| \ \|A\|_{\mathcal{S}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} < \infty \Big\},$$

where

$$\|A\|_{\mathcal{S}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} := \max\Big(\sup_{i\in\mathbb{Z}^d} \big\| \big(a(i,j)u(i,j)\big)_{j\in\mathbb{Z}^d} \big\|_p, \sup_{j\in\mathbb{Z}^d} \big\| \big(a(i,j)u(i,j)\big)_{i\in\mathbb{Z}^d} \big\|_p \Big)$$

[26, 38, 42, 44]. For $p = 1$, the class $\mathcal{S}_{1,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced by Schur [38] for weight matrices $u := (w(i)/w(j))_{i,j\in\mathbb{Z}^d}$ generated by positive sequences $w := (w(i))_{i\in\mathbb{Z}^d}$, and by Gröchenig and Leinert [26] for weight matrices $u := (v(i-j))_{i,j\in\mathbb{Z}^d}$ associated with positive functions v on $\mathbb{R}^d$. For $1 \le p \le \infty$, the class $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced by Sun for polynomial weights $u := ((1 + |i-j|_\infty)^\alpha)_{i,j\in\mathbb{Z}^d}$ with $\alpha > d(1-1/p)$ in [42] and for any weight matrix u in [44]. From the above definition of the Gröchenig-Schur class $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, the following inclusion follows:

$$\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \subset \mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \tag{2.3}$$

for any $1 \le p \le \infty$ and for any weight matrix u.

Remark 2.2. Let $1 \le p \le \infty$ and u be a weight matrix. Define the *Gohberg-Baskakov-Sjöstrand class* $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices by

$$\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ A := (a(i,j))_{i,j\in\mathbb{Z}^d} \ \Big| \ \|A\|_{\mathcal{C}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} < \infty \Big\},$$

where

$$\|A\|_{\mathcal{C}_{p,u}(\mathbb{Z}^d,\mathbb{Z}^d)} := \Big\| \Big(\sup_{i-j=k} \big(|a(i,j)|u(i,j)\big)\Big)_{k\in\mathbb{Z}^d} \Big\|_p$$

[6, 21, 26, 27, 33, 40, 44]. For $p = 1$ and the trivial weight matrix u_0 (i.e., $u_0(i,j) = 1$ for all $i, j \in \mathbb{Z}^d$), the class $\mathcal{C}_{1,u_0}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced by Gohberg, Kaashoek, and Woerdeman [21] as a generalization of the class of Laurent matrices associated with summable sequences. It was reintroduced by Sjöstrand [40] in considering an algebra of pseudo-differential operators. For $p = 1$ and nontrivial weight matrices $u := (v(i-j))_{i,j\in\mathbb{Z}^d}$ associated with positive functions v on $\mathbb{R}^d$, the class $\mathcal{C}_{1,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced and studied by Baskakov [6] and Kurbatov [33] independently, see also [26]. The above definition of the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ is given by Sun [44] for any $1 \le p \le \infty$ and any weight matrix u. From the definition of the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, we have the following inclusion:

$$\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \subset \mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \tag{2.4}$$

for any $1 \le p \le \infty$ and for any weight matrix u.

Remark 2.3. The inclusions (2.3) and (2.4) become equalities when $p = \infty$, i.e.,

$$\mathcal{B}_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) = \mathcal{C}_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) = \mathcal{S}_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) =: \mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d) \tag{2.5}$$

The class $\mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices is usually known as the *Jaffard class*, [6, 14, 24, 26, 30, 42, 44]. The Jaffard class $\mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d)$ with polynomial weight matrix $u := ((1 + |i-j|)^\alpha)_{i,j\in\mathbb{Z}^d}$ was introduced by Jaffard [30] in considering wavelets on an open domain. The Jaffard class $\mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d)$ with weight matrices $u := (v(i-j))_{i,j\in\mathbb{Z}^d}$ associated with positive functions v on $\mathbb{R}^d$ was introduced by Baskakov [6] independently, and later applied nontrivially in the study of localization of frames [26], adaptive computation [14], and nonuniform sampling [43].

For the class $\mathcal{B}_{p,u}$ of infinite matrices, we have

Proposition 2.4. *Let $\alpha \in \mathbb{C}$, $1 \le p \le \infty$, $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$ be a weight matrix, and let $A := ((a(i,j))_{i,j\in\mathbb{Z}^d}$ and $B := (b(i,j))_{i,j\in\mathbb{Z}^d}$ belong to $\mathcal{B}_{p,u}$. Then*

(i) $\|A + B\|_{\mathcal{B}_{p,u}} \le \|A\|_{\mathcal{B}_{p,u}} + \|B\|_{\mathcal{B}_{p,u}}$.
(ii) $\|\alpha A\|_{\mathcal{B}_{p,u}} = |\alpha|\|A\|_{\mathcal{B}_{p,u}}$.

(iii) $\|A^*\|_{\mathcal{B}_{p,u}} = \|A\|_{\mathcal{B}_{p,u}}$ *if* $u(i,j) = u(j,i)$ *for all* $i, j \in \mathbb{Z}^d$, *where* $A^* := \big(\overline{a(j,i)}\big)_{i,j\in\mathbb{Z}^d}$ *is the conjugate transpose of the matrix* A.

(iv) $\|A\|_{\mathcal{B}_{p,u}} \le \|B\|_{\mathcal{B}_{p,u}}$ *if* $|A| \le |B|$, *i.e.,* $|a(i,j)| \le |b(i,j)|$ *for all* $i, j \in \mathbb{Z}^d$.

All conclusions in the above proposition follow directly from (2.1) and (2.2). From the conclusions (i) and (ii) of the above proposition, we see that $\|\cdot\|_{\mathcal{B}_{p,u}}$ is a norm on the class $\mathcal{B}_{p,u}$ of infinite matrices. The properties in the conclusion (iv) is usually known as the *solidness* of the matrix norm $\|\cdot\|_{\mathcal{B}_{p,u}}$.

3. Algebraic properties

In this section, we establish some algebraic properties for the class $\mathcal{B}_{p,u}$ of infinite matrices and give a proof of Theorem 1.1.

Let us first recall the concept of a p-submultiplicative weight matrix u [26, 44, 45]. For $1 \le p \le \infty$, a weight matrix $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$ is said to be *p-submultiplicative* if there exists another weight matrix $v := (v(i,j))_{i,j\in\mathbb{Z}^d}$ such that

$$v(i,j) \ge 1 \quad \text{for all } i, j \in \mathbb{Z}^d, \tag{3.1}$$

$$u(i,j) \le u(i,k)v(k,j) + v(i,k)u(k,j) \quad \text{for all } i, j, k \in \mathbb{Z}^d, \tag{3.2}$$

and

$$C_p(v,u) := \Big\| \Big(\sup_{|i-j|_\infty \ge |k|_\infty} \big(v(i,j)(u(i,j))^{-1}\big) \Big) \Big\|_{p/(p-1)} < \infty. \tag{3.3}$$

For $p = 1$, we simply say that a weight matrix is *submultiplicative* instead of 1-submultiplicative. We call the weight matrix v satisfying (3.1), (3.2) and (3.3) a *companion weight matrix* of the p-submultiplicative weight matrix u. Denote by $C(u)$ the set of all companion weights of a p-submultiplicative weight matrix u, and define the *p-submultiplicative bound* $M_p(u)$ by $M_p(u) := \inf_{v\in C(u)} C_p(v,u)$. One may verify that $C(u)$ is a convex set and the infinimum of $C_p(v,u)$ in the set $C(u)$ can be attained for some companion weight matrix v. So from now on, except stated explicitly, we **always** assume that the companion weight v of a p-submultiplicative weight matrix u is the one satisfying

$$M_p(u) = C_p(v,u). \tag{3.4}$$

Remark 3.1. From the definitions of p-submultiplicative weight matrices on $\mathbb{Z}^d \times \mathbb{Z}^d$, we have the following:

(i) A p-submultiplicative weight matrix is q-submultiplicative for all $1 \le q \le p$.

(ii) A necessary condition for a weight matrix $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$ to be p-submultiplicative is $u(i,j) \le Cu(i,k)u(k,j)$ for all $i,j,k \in \mathbb{Z}^d$ and for some positive constant C. When $p=1$, the above necessary condition is also a sufficient condition [26].

(iii) Let $1 \le p \le \infty, \delta \in (0,1)$, and let α be a number with the property that $\alpha > d - d/p$ if $1 < p \le \infty$, and $\alpha \ge 0$ if $p = 1$. Then the Laurent matrices $p_\alpha := \big((1+|i-j|_\infty)^\alpha\big)_{i,j\in\mathbb{Z}^d}$ generated by the polynomial weight $(1+|x|_\infty)^\alpha$, and $e_\delta := \big(\exp(|i-j|_\infty^\delta)\big)_{i,j\in\mathbb{Z}^d}$ generated by the sub-exponential weight $\exp(|x|_\infty^\delta)$, are p-submultiplicative [44].

Now we state the main result of this section, an extension of Theorem 1.1.

Theorem 3.2. *Let $1 \le p \le \infty, 1 \le q < \infty$, u be a p-submultiplicative weight matrix with the p-submultiplicative bound $M_p(u)$, and let w be a discrete A_q-weight with the A_q-bound $A_q(w)$. Then the following statements hold.*

(i) *If v is a companion weight matrix of the p-submultiplicative weight matrix u, then*

$$\|AB\|_{\mathcal{B}_{p,u}} \le 2^{2/p}5^{(d-1)/p}\big(\|A\|_{\mathcal{B}_{p,u}}\|B\|_{\mathcal{B}_{1,v}} + \|A\|_{\mathcal{B}_{1,v}}\|B\|_{\mathcal{B}_{p,u}}\big) \tag{3.5}$$

for all $A, B \in \mathcal{B}_{p,u}$.

(ii) *$\mathcal{B}_{p,u}$ is (and hence $\mathcal{B}$ is also) an algebra. Moreover*

$$\|AB\|_{\mathcal{B}_{p,u}} \le 2^{1+2/p}5^{(d-1)/p}M_p(u)\|A\|_{\mathcal{B}_{p,u}}\|B\|_{\mathcal{B}_{p,u}} \quad \text{for all } A, B \in \mathcal{B}_{p,u}. \tag{3.6}$$

(iii) *$\mathcal{B}_{p,u}$ is a subalgebra of $\mathcal{B}$. Moreover*

$$\|A\|_{\mathcal{B}} \le M_p(u)\|A\|_{\mathcal{B}_{p,u}} \quad \text{for all } A \in \mathcal{B}_{p,u}. \tag{3.7}$$

(iv) *$\mathcal{B}_{p,u}$ is (and hence $\mathcal{B}$ is also) a subalgebra of $\mathcal{B}(\ell^q_w)$. Moreover*

$$\|Ac\|_{q,w} \le 2^{2d}3^{d/q}(A_q(w))^{1/q}M_p(u)\|A\|_{\mathcal{B}_{p,u}}\|c\|_{q,w} \tag{3.8}$$

for all $A \in \mathcal{B}_{p,u}$ and $c \in \ell^q_w$.

Before we give the proof of the above theorem, let us next make some remarks on the unital Banach algebra property of the algebra $\mathcal{B}_{p,u}$, on the equality of spectral radii in the algebras $\mathcal{B}_{p,u}$ and $\mathcal{B}_{1,v}$, and on the inclusion $\mathcal{B}_{p,u} \subset \mathcal{B}(\ell^q_w)$.

Remark 3.3. For $1 \le p \le \infty$ and a p-submultiplicative weight matrix $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$, following the standard procedure [20, 36] we define $\|A\|'_{\mathcal{B}_{p,u}} := \sup_{\|B\|_{\mathcal{B}_{p,u}}=1}\|AB\|_{\mathcal{B}_{p,u}}$ for $A \in \mathcal{B}_{p,u}$. Then

$$\|AB\|'_{\mathcal{B}_{p,u}} \le \|A\|'_{\mathcal{B}_{p,u}}\|B\|'_{\mathcal{B}_{p,u}} \quad \text{for all } A, B \in \mathcal{B}_{p,u}.$$

If the weight matrix u further satisfies

$$M := \sup_{i\in\mathbb{Z}^d} u(i,i) < \infty, \tag{3.9}$$

then the identity matrix I belongs to $\mathcal{B}_{p,u}$, and the norms $\|\cdot\|_{\mathcal{B}_{p,u}}$ and $\|\cdot\|'_{\mathcal{B}_{p,u}}$ on $\mathcal{B}_{p,u}$ are equivalent to each other, because

$$M^{-1}\|A\|_{\mathcal{B}_{p,u}} \le \|A\|'_{\mathcal{B}_{p,u}} \le 2^{1+2/p}5^{(d-1)/p}M_p(u)\|A\|_{\mathcal{B}_{p,u}} \quad \text{for all } A\in\mathcal{B}_{p,u}$$

by the conclusion (ii) of Theorem 3.2 and the fact that $\|I\|_{\mathcal{B}_{p,u}} = M$. Therefore if $1 \le p \le \infty$ and u is a p-submultiplicative weight matrix satisfying (3.9), then the class $\mathcal{B}_{p,u}$ of infinite matrices endowed with the norm $\|\cdot\|'_{\mathcal{B}_{p,u}}$ becomes a unital Banach algebra.

Remark 3.4. Let $1 \le p \le \infty$, u be a p-submultiplicative weight matrix satisfying (3.9), and v be its companion weight matrix. If the companion weight matrix v is submultiplicative, then both $\mathcal{B}_{p,u}$ and $\mathcal{B}_{1,v}$ are algebras by the conclusion (ii) of Theorem 3.2, and $\mathcal{B}_{p,u}$ is a subalgebra of $\mathcal{B}_{1,v}$ since

$$\|A\|_{\mathcal{B}_{1,v}} \le C_p(v,u)\|A\|_{\mathcal{B}_{p,u}} \quad \text{for all } A\in\mathcal{B}_{p,u}. \tag{3.10}$$

Applying (3.10) with A replaced by A^n and then taking n-th roots and the limit as $n\to\infty$ yields

$$\rho_{\mathcal{B}_{1,v}}(A) := \limsup_{n\to\infty}(\|A^n\|_{\mathcal{B}_{1,v}})^{1/n} \le \limsup_{n\to\infty}(\|A^n\|_{\mathcal{B}_{p,u}})^{1/n} =: \rho_{\mathcal{B}_{p,u}}(A).$$

From the conclusion (i) of Theorem 3.2 it follows that

$$\|A^{2n}\|_{\mathcal{B}_{p,u}} \le 2^{1+2/p}5^{(d-1)/p}\|A^n\|_{\mathcal{B}_{p,u}}\|A^n\|_{\mathcal{B}_{1,v}}.$$

Taking n-th roots on both sides of the above inequality and then letting $n\to\infty$ lead to the inequality $\rho_{\mathcal{B}_{p,u}}(A) \le \rho_{\mathcal{B}_{1,v}}(A)$. This implies that if u is a p-submultiplicative weight matrix and its companion weight matrix v is submultiplicative, then the spectral radii $\rho_{\mathcal{B}_{p,u}}(A)$ and $\rho_{\mathcal{B}_{1,v}}(A)$ are the same for any $A\in\mathcal{B}_{p,u}$, i.e., $\rho_{\mathcal{B}_{1,v}}(A) = \rho_{\mathcal{B}_{p,u}}(A)$ for all $A\in\mathcal{B}_{p,u}$. The above procedure to establish the equality of spectral radii in the algebras $\mathcal{B}_{p,u}$ and $\mathcal{B}_{1,v}$ from the inequality in the conclusion (i) of Theorem 3.2 is known as *Brandenburg's trick* [12, 24]. Another technique to prove the equality of spectral radii in two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with the same unit element is by showing that

$$\|A\|_{\mathcal{A}_2} \le C\|A\|_{\mathcal{A}_1} \tag{3.11}$$

and

$$\|A^2\|_{\mathcal{A}_1} \le C\|A\|_{\mathcal{A}_1}^{1+\theta}\|A\|_{\mathcal{A}_2}^{1-\theta} \quad \text{for all } A\in\mathcal{A}_1, \tag{3.12}$$

where $\|\cdot\|_{\mathcal{A}_1}$ and $\|\cdot\|_{\mathcal{A}_2}$ are norms in the algebra $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, and where $C\in(0,\infty)$ and $\theta\in[0,1)$ are constants independent of

$A \in \mathcal{A}$. The estimates in (3.11) and (3.12) for $\mathcal{A}_2 = \mathcal{B}(\ell^2)$ and $\mathcal{A}_1 = \mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ or $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ are established in [42, 44], while those for $\mathcal{A}_2 = \mathcal{B}(\ell^2)$ and $\mathcal{A}_1 = \mathcal{B}_{p,u}, 1 \le p \le \infty$, are given in Lemma 5.3.

Remark 3.5. The conclusion (iv) of Theorem 3.2 about the boundedness of an infinite matrix in $\mathcal{B}$ on the weighted sequence space ℓ^q_w is a simplified discrete version of the second conclusion in [41, Proposition 2 of Chapter 10]. The reader may refer to [27, Lemma 3.1] for a general result on the boundedness of an infinite matrix on sequence spaces.

We conclude this section by giving the proof of Theorem 3.2.

Proof of Theorem 3.2. **(i)**: Let $1 \le p \le \infty$, u be a p-submultiplicative weight matrix, and let v be a companion weight matrix of the weight matrix u. Take $A := (a(i,j))_{i,j\in\mathbb{Z}^d}$ and $B := (b(i,j))_{i,j\in\mathbb{Z}^d}$ in $\mathcal{B}_{p,u}$, and write $AB := (c(i,j))_{i,j\in\mathbb{Z}^d}$. Then it follows from (3.2) that

$$\begin{aligned} |c(i,j)|u(i,j) &= \Big|\sum_{k\in\mathbb{Z}^d} a(i,k)b(k,j)\Big| u(i,j) \\ &\le \sum_{k\in\mathbb{Z}^d} |a(i,k)|u(i,k)|b(k,j)|v(k,j) \\ &\quad + \sum_{k\in\mathbb{Z}^d} |a(i,k)|v(i,k)|b(k,j)|u(k,j) \quad \text{for all } i,j\in\mathbb{Z}^d. \end{aligned} \tag{3.13}$$

For $1 \le p < \infty$, we obtain from (3.13) that

$$
\begin{aligned}
& \sum_{k\in\mathbb{Z}^d} |a(i,k)|u(i,k)|b(k,j)|v(k,j) \\
\le\ & \Big(\sum_{k'\in\mathbb{Z}^d} (|a(i,k')|u(i,k'))^p |b(k',j)|v(k',j)\Big)^{1/p} \\
& \quad \times \Big(\sum_{k''\in\mathbb{Z}^d} |b(k'',j)|v(k'',j)\Big)^{(p-1)/p} \\
\le\ & \big(\|B\|_{\mathcal{B}_{1,v}}\big)^{(p-1)/p} \Big\{ \Big(\sum_{k'\in\mathbb{Z}^d} |b(k',j)|v(k',j)\Big) \\
& \qquad \times \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|a(i',j')|u(i',j')\big)^p\Big) \\
& \quad + \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|b(i',j')|v(i',j')\big)\Big) \\
& \qquad \times \Big(\sum_{k'\in\mathbb{Z}^d} \big(|a(i,k')|u(i,k')\big)^p\Big)\Big\}^{1/p} \\
\le\ & \big(\|B\|_{\mathcal{B}_{1,v}}\big)^{(p-1)/p} \Big\{ \|B\|_{\mathcal{B}_{1,v}} \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|a(i',j')|u(i',j')\big)^p\Big) \\
& \quad + \big(\|A\|_{\mathcal{B}_{p,u}}\big)^p \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|b(i',j')|v(i',j')\big)\Big)\Big\}^{1/p},
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{k\in\mathbb{Z}^d} |a(i,k)|v(i,k)|b(k,j)|u(k,j) \\
\le\ & \big(\|A\|_{\mathcal{B}_{1,v}}\big)^{(p-1)/p} \Big\{ \|A\|_{\mathcal{B}_{1,v}} \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|b(i',j')|u(i',j')\big)^p\Big) \\
& \quad + \big(\|B\|_{\mathcal{B}_{p,u}}\big)^p \Big(\sup_{|i'-j'|_\infty \ge |i-j|_\infty/2} \big(|a(i',j')|v(i',j')\big)\Big)\Big\}^{1/p}.
\end{aligned}
$$

Combining the above two estimates with (3.13) leads to

$$\begin{aligned}
\|AB\|_{\mathcal{B}_{p,u}} &= \Big\|\Big(\sup_{|i-j|_\infty\ge|k|_\infty}\big(|c(i,j)|u(i,j)\big)\Big)_{k\in\mathbb{Z}^d}\Big\|_p\\
&\le \big(\|B\|_{\mathcal{B}_{1,v}}\big)^{(p-1)/p}\Big\{\|B\|_{\mathcal{B}_{1,v}}\\
&\qquad\times\Big\|\Big(\sup_{|i-j|_\infty\ge|k|_\infty/2}\big(|a(i',j')|u(i',j')\big)^p\Big)_{k\in\mathbb{Z}^d}\Big\|_1\\
&\qquad+\big(\|A\|_{\mathcal{B}_{p,u}}\big)^p\Big\|\Big(\sup_{|i-j|_\infty\ge|k|_\infty/2}\big(|b(i',j')|v(i',j')\big)\Big)_{k\in\mathbb{Z}^d}\Big\|_1\Big\}^{1/p}\\
&\quad+\big(\|A\|_{\mathcal{B}_{1,v}}\big)^{(p-1)/p}\Big\{\|A\|_{\mathcal{B}_{1,v}}\\
&\qquad\times\Big\|\Big(\sup_{|i-j|_\infty\ge|k|_\infty/2}\big(|b(i',j')|u(i',j')\big)^p\Big)_{k\in\mathbb{Z}^d}\Big\|_1\\
&\qquad+\big(\|B\|_{\mathcal{B}_{p,u}}\big)^p\Big\|\Big(\sup_{|i-j|_\infty\ge|k|_\infty/2}\big(|a(i',j')|v(i',j')\big)\Big)_{k\in\mathbb{Z}^d}\Big\|_1\Big\}^{1/p}\\
&\le 2^{2/p}5^{(d-1)/p}\Big(\|A\|_{\mathcal{B}_{p,u}}\|B\|_{\mathcal{B}_{1,v}}+\|A\|_{\mathcal{B}_{1,v}}\|B\|_{\mathcal{B}_{p,u}}\Big),
\end{aligned}$$

where we have used the fact that

$$\Big\|\Big(\sup_{|i-j|_\infty\ge|k|_\infty/N}|a(i,j)|\Big)\Big\|_1\le N(2N+1)^{d-1}\Big\|\Big(\sup_{|i-j|_\infty\ge|k|_\infty}|a(i,j)|\Big)\Big\|_1 \tag{3.14}$$

for any integer $N\ge 1$ and $A:=(a(i,j))\in\mathcal{B}$. This proves (3.5) for $1\le p<\infty$.

For $p=\infty$, it follows from (3.13) that

$$\|AB\|_{\mathcal{B}_{\infty,u}}\le\|A\|_{\mathcal{B}_{\infty,u}}\|B\|_{\mathcal{B}_{1,v}}+\|A\|_{\mathcal{B}_{1,v}}\|B\|_{\mathcal{B}_{\infty,u}}.$$

Hence (3.5) for $p=\infty$ is proved.

(ii) Let v be the companion weight matrix of the p-submultiplicative weight u that satisfies (3.1)–(3.4). Then

$$\|A\|_{\mathcal{B}_{1,v}}\le M_p(u)\|A\|_{\mathcal{B}_{p,u}}\quad\text{for all } A\in\mathcal{B}_{p,u}, \tag{3.15}$$

because

$$\begin{aligned}
\sup_{|i-j|_\infty\ge|k|_\infty}|a(i,j)|v(i,j) &\le \Big(\sup_{|i'-j'|_\infty\ge|k|_\infty}|a(i',j')|u(i',j')\Big)\\
&\quad\times\Big(\sup_{|i'-j'|_\infty\ge|k|_\infty}v(i',j')(u(i',j'))^{-1}\Big)
\end{aligned}$$

hold for all $k\in\mathbb{Z}^d$. Combining (3.5) and (3.15) proves (3.6).

(iii) Let v be the companion weight matrix of the p-submultiplicative weight u that satisfies (3.1)–(3.4). Then

$$\|A\|_{\mathcal{B}} \le \|A\|_{\mathcal{B}_{1,v}} \quad \text{for all } A \in \mathcal{B}_{1,v}$$

by (3.1) for the weight matrix v. This together with (3.15) gives (3.7) and hence proves the conclusion (iii).

(iv) By (iii), it suffices to prove

$$\|Ac\|_{q,w} \le 2^{2d} 3^{d/q} (A_q(w))^{1/q} \|A\|_{\mathcal{B}} \|c\|_{q,w} \tag{3.16}$$

for all $A := (a(i,j))_{i,j\in\mathbb{Z}^d} \in \mathcal{B}$ and $c \in \ell^q_w$. Set $h(n) := \sup_{|i-j|_\infty \ge n} |a(i,j)|$. Then $\{h(n)\}_{n=0}^\infty$ is a decreasing sequence, i.e., $h(n+1) \le h(n)$ for all $n \ge 0$, and

$$\begin{aligned}
\sum_{l=1}^\infty h(2^{l-1}) 2^{(l+1)d} & \le 2^{2d} h(1) + 2^{d+2} \sum_{l=2}^\infty \Big(\sum_{2^{l-2} < s \le 2^{l-1}} h(s) \Big) 2^{l(d-1)} \\
& \le 2^{2d} h(1) + 2^{3d} \sum_{s=2}^\infty h(s) s^{d-1} \\
& \le 2^{2d} h(1) + 2^{2d} d^{-1} \sum_{s=2}^\infty \sum_{k\in\mathbb{Z}^d \text{ with } |k|_\infty = s} h(|k|_\infty) \\
& \le 2^{2d} (\|A\|_{\mathcal{B}} - h(0)).
\end{aligned}$$

For $1 < q < \infty$ and a discrete A_q-weight w,

$$\begin{aligned}
\|Ac\|_{q,w} & \le \Big\{ \sum_{i\in\mathbb{Z}^d} \Big(\sum_{j\in\mathbb{Z}^d} h(|i-j|_\infty) |c(j)| \Big)^q w(i) \Big\}^{1/q} \\
& \le h(0) \Big\{ \sum_{i\in\mathbb{Z}^d} |c(i)|^q w(i) \Big\}^{1/q} + \Big\{ \sum_{i\in\mathbb{Z}^d} w(i) \Big(\sum_{l=1}^\infty h(2^{l-1}) 2^{(l+1)d} \Big)^{q-1} \\
& \quad \times \Big(\sum_{l=1}^\infty h(2^{l-1}) 2^{-(l+1)d(q-1)} \Big(\sum_{2^{l-1} \le |i-j|_\infty < 2^l} |c(j)| \Big)^q \Big) \Big\}^{1/q}.
\end{aligned}$$

Thus

$$\begin{aligned}\|Ac\|_{q,w} &\le h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{\mathcal{B}} - h(0))^{(q-1)/q}\\ &\quad \times\Big\{\sum_{i\in\mathbb{Z}^d}\sum_{l=1}^{\infty} w(i)h(2^{l-1})2^{-(l+1)d(q-1)}\\ &\quad \times\Big(\sum_{2^{l-1}\le|i-j|_\infty<2^l}|c(j)|^q w(j)\Big)\\ &\quad \times\Big(\sum_{2^{l-1}\le|i-j'|_\infty<2^l}(w(j'))^{-1/(q-1)}\Big)^{q-1}\Big\}^{1/q}.\end{aligned}$$

This together with the discrete A_q-weight assumption leads to

$$\begin{aligned}\|Ac\|_{q,w} &\le h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{\mathcal{B}} - h(0))^{(q-1)/q}(A_q(w))^{1/q}\\ &\quad \times\Big\{\sum_{l=1}^{\infty} h(2^{l-1})2^{(l+1)d}\sum_{i\in\mathbb{Z}^d}\frac{w(i)}{\sum_{|i-j'|_\infty<2^l} w(j')}\\ &\quad \times\Big(\sum_{2^{l-1}\le|i-j|_\infty<2^l}|c(j)|^q w(j)\Big)\Big\}^{1/q}\\ &\le h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{\mathcal{B}} - h(0))^{(q-1)/q}(A_q(w))^{1/q}\\ &\quad \times\Big\{\sum_{l=1}^{\infty} h(2^{l-1})2^{(l+1)d}\Big(\sum_{j\in\mathbb{Z}^d}|c(j)|^q w(j)\\ &\quad \Big(\sum_{\epsilon\in\{-1,0,1\}^d}\sum_{|i-j-\epsilon 2^{l-1}|_\infty<2^{l-1}}\frac{w(i)}{\sum_{|i-j'|_\infty<2^l} w(j')}\Big)\Big)\Big\}^{1/q}\\ &\le 2^{2d}3^{d/p}(A_q(w))^{1/q}\|A\|_{\mathcal{B}}\|c\|_{q,w},\end{aligned}$$

and hence (3.8) for $1<q<\infty$ is established.

The conclusion (3.8) for $q=1$ can be proved by similar arguments. We omit the details here. □

4. ℓ^q_w-stability

In this section, we prove the following theorem (a slight generalization of Theorem 1.2) and Corollary 1.3. We also provide a characterization of the ℓ^q_w-stability of a Laurent matrix in $\mathcal{B}$.

Theorem 4.1. *Let $1\le p\le\infty$, $1\le q,q'<\infty$, let the weight matrix u be p-submultiplicative, and w,w' be discrete A_q-weight and $A_{q'}$-weight respectively. If $A\in\mathcal{B}_{p,u}$ has ℓ^q_w-stability, then A has $\ell^{q'}_{w'}$-stability.*

As the trivial weight w_0 (i.e. $w_0(i) = 1$ for all $i \in \mathbb{Z}^d$) is a discrete A_q-weight for any $1 \le q < \infty$, we have the following corollary of Theorem 4.1.

Corollary 4.2. *If $A \in \mathcal{B}$ has ℓ^q-stability for some $1 \le q < \infty$, then A has $\ell^{q'}$-stability for all $1 \le q' < \infty$.*

We remark that similar results about ℓ^q-stability for different exponents $q \in [1, \infty]$ are established by Aldroubi, Baskakov and Krishtal [1] for infinite matrices in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{1,p_\alpha}(\mathbb{Z}^d, \mathbb{Z}^d)$ with $\alpha > (d+1)^2$, by Tessera [50] for $\alpha > 0$, and by Shin and Sun [39] for $\alpha \ge 0$, where $p_\alpha = ((1 + |i - j|_\infty)^\alpha)_{i,j \in \mathbb{Z}^d}$.

The ℓ^q_w-stability is one of the basic assumptions for infinite matrices arising in many fields of mathematics (see [2, 4, 25, 29, 30, 40, 43, 44] for a sample of papers), but little is known about practical criteria for the ℓ^q_w-stability of an infinite matrix, see [47] for the diagonal-blocks-dominated criterion for the ℓ^2-stability of infinite matrices in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$. As an application of Theorem 1.2, we have the following characterization of the ℓ^q_w-stability of a Laurent matrix in $\mathcal{B}$.

Corollary 4.3. *Let $1 \le q < \infty$, $A := (a(i-j))_{i,j \in \mathbb{Z}^d}$ be a Laurent matrix in $\mathcal{B}$, and let w be a discrete A_q-weight. Then A has ℓ^q_w-stability if and only if $\hat{a}(\xi) := \sum_{n \in \mathbb{Z}^d} a(n) e^{-\sqrt{-1}\, n\xi} \ne 0$ for all $\xi \in \mathbb{R}^d$.*

To prove Theorem 4.1, we recall a characterization of discrete A_q-weights.

Lemma 4.4. [19, 41] *Let $1 \le q < \infty$. Then $w := (w(i))_{i \in \mathbb{Z}^d}$ is a discrete A_q-weight with the A_q-bound $A_q(w)$ if and only if*

$$\begin{aligned} &\Big(N^{-d} \sum_{i \in a + [0, N-1]^d} |c(i)|\Big)^q \Big(N^{-d} \sum_{i \in a+[0,N-1]^d} w(i)\Big) \\ \le\ & A_q(w) N^{-d} \sum_{i \in a+[0,N-1]^d} |c(i)|^q w(i) \end{aligned} \tag{4.1}$$

hold for all $a \in \mathbb{Z}^d, 1 \le N \in \mathbb{Z}$ and sequences $c := (c(i))_{i \in \mathbb{Z}^d}$.

To prove Theorem 4.1, we need a technical lemma about estimating a bounded sequence c via the sequence Ac, which will also be used later in the proof of Theorem 1.4. Similar estimate is given in [40] when the infinite matrix A belongs to the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$ and has ℓ^p_w-stability for the trivial weight $w \equiv 1$.

Lemma 4.5. *Let $1 \le q < \infty$, and w be a discrete A_q-weight. If $A \in \mathcal{B}$ has ℓ^q_w-stability, then there exists a nonnegative sequence $\{g(i)\}_{i\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$ such that*

$$\sum_{k\in\mathbb{Z}^d}\Big(\sup_{|i|_\infty\ge|k|_\infty} g(i)\Big)<\infty \tag{4.2}$$

and

$$|c(i)|\le\sum_{j\in\mathbb{Z}^d} g(i-j)|(Ac)(j)|,\ \ i\in\mathbb{Z}^d, \tag{4.3}$$

where $c\in\ell^\infty$.

Proof. Without loss of generality, we assume that

$$\|c\|_{q,w}\le\|Ac\|_{q,w}\quad\text{for all } c\in\ell^q_w. \tag{4.4}$$

Let $h(x)=\min(\max(2-|x|_\infty,0),1)$ and N be a sufficiently large integer chosen later. Define linear operators $\Psi^N_n, n\in N\mathbb{Z}^d$, on ℓ^q_w by

$$\Psi^N_n c:=\Big(h\big(\frac{j-n}{N}\big)c(j)\Big)_{j\in\mathbb{Z}^d}\quad\text{for } c:=(c(j))_{j\in\mathbb{Z}^d}\in\ell^q_w.$$

Then for $c:=(c(j))_{j\in\mathbb{Z}^d}\in\ell^q_w$ and $|n-n'|_\infty\le 8N$,

$$\begin{aligned}
&\|(\Psi^N_nA-A\Psi^N_n)\Psi^N_{n'}c\|_{q,w}\\
=\ &\Big\{\sum_{i\in\mathbb{Z}^d}\Big|\sum_{j\in\mathbb{Z}^d}\Big(h\big(\frac{i-n}{N}\big)-h\big(\frac{j-n}{N}\big)\Big)\\
&\quad\times a(i,j)h\big(\frac{j-n'}{N}\big)c(j)\Big|^q w(i)\Big\}^{1/q}\\
\le\ & N^{-1/2}\Big\{\sum_{i\in\mathbb{Z}^d}\Big(\sum_{|i-j|_\infty\le\sqrt{N}}|a(i,j)||c(j)|\Big)^q w(i)\Big\}^{1/q}\\
&+\Big\{\sum_{i\in\mathbb{Z}^d}\Big(\sum_{|i-j|_\infty>\sqrt{N}}|a(i,j)||c(j)|\Big)^q w(i)\Big\}^{1/q}\\
\le\ &\Big\{2^{2d+2d/q}N^{-1/2}(A_q(w))^{1/q}\|A\|_{\mathcal{B}}+2^{3d+2d/q+1}(A_q(w))^{1/q}\\
&\quad\times\Big(\sum_{|k|_\infty\ge\sqrt{N}/2}\sup_{|i-j|_\infty\ge|k|_\infty}|a(i,j)|\Big)\Big\}\|c\|_{q,w},
\end{aligned}\tag{4.5}$$

where the last inequality follows from (3.16) and the following estimate:

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \sup_{|i-j|_\infty\ge \max(|k|_\infty,\sqrt{N})} |a(i,j)|\\
\le\ & (2\sqrt{N}+1)^d \sup_{|i-j|_\infty\ge\sqrt{N}} |a(i,j)| + \sum_{|k|_\infty>\sqrt{N}} \sup_{|i-j|_\infty\ge|k|_\infty} |a(i,j)|\\
\le\ & 2^{d+1} \sum_{|k|_\infty\ge\sqrt{N}/2} \sup_{|i-j|_\infty\ge|k|_\infty} |a(i,j)|.
\end{aligned}$$

Similarly for $c := (c(j))_{j\in\mathbb{Z}^d} \in \ell^q_w$ and $|n-n'|_\infty > 8N$,

$$\begin{aligned}
& \|(\Psi_n^N A - A\Psi_n^N)\Psi_{n'}^N c\|_{q,w}\\
=\ & \Big(\sum_{i\in\mathbb{Z}^d}\Big|\sum_{j\in\mathbb{Z}^d} h\big(\frac{i-n}{N}\big)a(i,j)h\big(\frac{j-n'}{N}\big)c(j)\Big|^q w(i)\Big)^{1/q}\\
\le\ & \Big(\sup_{|i'-j'|_\infty\ge|n-n'|_\infty/2} |a(i',j')|\Big)\Big(\sum_{|i-n|_\infty<2N}\Big(\sum_{|j-n'|_\infty<2N}|c(j)|\Big)^q w(i)\Big)^{1/q}\\
\le\ & 2^{2d}N^d(A_q(w))^{1/q}\Big(\sup_{|i'-j'|_\infty\ge|n-n'|_\infty/2}|a(i',j')|\Big)\\
& \times\Big(\frac{\sum_{|i'-n|_\infty<2N} w(i')}{\sum_{|i'-n'|_\infty<2N} w(i')}\Big)^{1/q}\|c\|_{q,w}. \qquad (4.6)
\end{aligned}$$

Define

$$\alpha_n := \sum_{|i'-n|_\infty<2N} w(i'), \quad n\in N\mathbb{Z}^d$$

and the linear operator Φ_N on ℓ^p_w by

$$\Phi_N c := \Big(\Big(\sum_{n\in N\mathbb{Z}^d}\Big(h\big(\frac{j-n}{N}\big)\Big)^2\Big)^{-1} c(j)\Big)_{j\in\mathbb{Z}^d} \quad \text{for } c := (c(j))_{j\in\mathbb{Z}^d}\in\ell^p_w.$$

Then for all $n'\in N\mathbb{Z}^d$ with $|n-n'|\le 8N$,

$$\alpha_n \le \sum_{|i'-n'|_\infty<10N} w(i') \le 6^{dq}A_q(w)\alpha_{n'} \qquad (4.7)$$

by (4.1), and

$$\|\Phi_N c\|_{q,w} \le \|c\|_{q,w} \quad \text{for all } c\in\ell^q_w. \qquad (4.8)$$

Note that $\Psi_n^N c\in\ell^p_w$ for any $c\in\ell^\infty$ and $n\in N\mathbb{Z}^d$, and

$$\|\Psi_n^N c\|_{q,w} \le \alpha_n^{1/q}\|c\|_\infty,\ n\in N\mathbb{Z}^d. \qquad (4.9)$$

Then for $c\in\ell^\infty$, combining (4.4), (4.5), (4.6), (4.7), and (4.8) leads to

$$
\begin{aligned}
(4.10)\quad & \alpha_n^{-1/q}\|\Psi_n^N c\|_{q,w}\le \alpha_n^{-1/q}\|A\Psi_n^N c\|_{q,w}\\
\le\ & \alpha_n^{-1/q}\|\Psi_n^N Ac\|_{q,w}+\alpha_n^{-1/q}\|(\Psi_n^N A-A\Psi_n^N)c\|_{q,w}\\
\le\ & \alpha_n^{-1/q}\|\Psi_n^N Ac\|_{q,w}+\alpha_n^{-1/q}\sum_{n'\in N\mathbb{Z}^d}\big\|(\Psi_n^N A-A\Psi_n^N)\Psi_{n'}^N\Phi_N\Psi_{n'}^N c\big\|_{q,w}\\
\le\ & \alpha_n^{-1/q}\|\Psi_n^N Ac\|_{q,w}+2^{3d+2d/q}3^d(A_q(w))^{2/q}\sum_{|n'-n|_\infty\le 8N}\alpha_{n'}^{-1/q}\|\Psi_{n'}^N c\|_{q,w}\\
& \times\Big\{\Big(N^{-1/2}\|A\|_{\mathcal{B}}+2^{d+1}\sum_{|k|_\infty\ge\sqrt{N}/2}\ \sup_{|i'-j'|_\infty\ge|k|_\infty}|a(i',j')|\Big)\Big\}\\
& +\sum_{|n'-n|_\infty>8N}\alpha_{n'}^{-1/q}\|\Psi_{n'}^N c\|_{q,w}\\
& \times\Big\{2^{2d}N^d(A_q(w))^{1/q}\Big(\sup_{|i'-j'|_\infty\ge|n-n'|_\infty/2}|a(i',j')|\Big)\Big\}\\
=:\ & \alpha_n^{-1/q}\|\Psi_n^N Ac\|_{q,w}+\sum_{n'\in N\mathbb{Z}^d}V_N(n-n')\alpha_{n'}^{-1/q}\|\Psi_{n'}^N c\|_{q,w}.
\end{aligned}
$$

Define sequences $V_N^l:=(V_N^l(n))_{n\in N\mathbb{Z}^d}, l\ge 1$, as follows:

$$
\begin{cases}
V_N^l(n):=V_N(n) & \text{if } l=1 \text{ and } n\in N\mathbb{Z}^d,\\
V_N^l(n):=\sum_{n'\in N\mathbb{Z}^d}V_N(n-n')V_N^{l-1}(n') & \text{if } l\ge 2 \text{ and } n\in N\mathbb{Z}^d.
\end{cases}
$$

Then for $c\in\ell^\infty$, applying (4.10) repeatedly yields

$$
\begin{aligned}
& \alpha_n^{-1/q}\|\Psi_n^N c\|_{q,w}\\
\le\ & \alpha_n^{-1/q}\|\Psi_n^N Ac\|_{q,w}+\sum_{l=1}^{l_0}\sum_{n'\in N\mathbb{Z}^d}V_N^l(n-n')\alpha_{n'}^{-1/q}\|\Psi_{n'}^N Ac\|_{q,w}\\
(4.11)\quad & +\sum_{n'\in N\mathbb{Z}^d}V_N^{l_0+1}(n-n')\alpha_{n'}^{-1/q}\|\Psi_{n'}^N c\|_{q,w},\qquad l_0\ge 1.
\end{aligned}
$$

Set

$$\epsilon_N^l:=\sum_{k\in N\mathbb{Z}^d}\sup_{|n|_\infty\ge|k|_\infty}|V_N^l(n)|.$$

Inductively for $l\ge 2$,

$$
\begin{aligned}
\epsilon_N^l\ \le\ & \epsilon_N^{l-1}\sum_{k\in N\mathbb{Z}^d}\sup_{|n|_\infty\ge|k|_\infty/2}|V_N(n)|\\
& +\epsilon_N^1\sum_{k\in N\mathbb{Z}^d}\sup_{|n|_\infty\ge|k|_\infty/2}|V_N^{l-1}(n)|\le 5^d\epsilon_N^1\epsilon_N^{l-1},
\end{aligned}
$$

where we have used (3.14) to obtain the last inequality. This shows that

$$\epsilon_N^l \le (5^d \epsilon_N^1)^l \quad \text{for all } l \ge 1. \tag{4.12}$$

Note that

$$\begin{aligned}
\epsilon_N^1 &\le 2^{4d+2d/q}3^{3d}(A_q(w))^{2/q}\Big\{N^d\Big(\sup_{|i'-j'|_\infty>4N}|a(i',j')|\Big)\\
&\quad +2^{d+1}\Big(\sum_{k'\in\mathbb{Z}^d,\ |k'|_\infty\ge\sqrt{N}/2}\ \sup_{|i'-j'|_\infty\ge|k'|_\infty}|a(i',j')|\Big)\\
&\quad +N^{-1/2}\|A\|_{\mathcal{B}}\Big\}+2^{2d}(A_q(w))^{1/q}\\
&\quad \times\Big\{\sum_{|k|_\infty>8N,k\in N\mathbb{Z}^d}N^d\Big(\sup_{|i'-j'|_\infty\ge|k|_\infty/2}|a(i',j')|\Big)\Big\}\\
&\le 2^{4d+2d/q}3^{3d}(A_q(w))^{2/q}\Big\{N^{-1/2}\|A\|_{\mathcal{B}}\\
&\quad +2^{d+2}\Big(\sum_{k'\in\mathbb{Z}^d,\ |k'|_\infty\ge\sqrt{N}/2}\ \sup_{|i'-j'|_\infty\ge|k'|_\infty}|a(i',j')|\Big)\Big\}\\
&\quad +2^{2d+1}(A_q(w))^{1/q}\sum_{k'\in\mathbb{Z}^d,\ |k'|>7N}\Big(\sup_{|i'-j'|_\infty\ge7|k'|_\infty/16}|a(i',j')|\Big)\\
&\le 2^{6d}3^{3d}(A_q(w))^{2/q}N^{-1/2}\|A\|_{\mathcal{B}}+2^{7d+3}3^{3d}(A_q(w))^{2/q}\\
&\quad\times\Big(\sum_{k'\in\mathbb{Z}^d,\ |k'|_\infty\ge\sqrt{N}/2}\big(\sup_{|i'-j'|_\infty\ge|k'|_\infty}|a(i',j')|\big)\Big)\\
&\to 0 \quad \text{as } N\to+\infty
\end{aligned}$$

by the assumption $A\in\mathcal{B}$. Let N be the integer chosen sufficiently large so that

$$\epsilon_N^1 < 5^{-d}. \tag{4.13}$$

Taking the limit as $l_0\to\infty$ in (4.11), and using (4.9), (4.12) and (4.13) lead to

$$\begin{aligned}
\alpha_n^{-1/q}\|\Psi_n^N c\|_{q,w} &\le \alpha_n^{-1/q}\|\Psi_n^N Ac\|_{q,w}\\
&\quad+\sum_{n'\in N\mathbb{Z}^d}\Big(\sum_{l=1}^{\infty}V_N^l(n-n')\Big)\alpha_{n'}^{-1/q}\|\Psi_{n'}^N Ac\|_{q,w}\\
&=: \sum_{n'\in N\mathbb{Z}^d}W_N(n-n')\alpha_{n'}^{-1/q}\|\Psi_{n'}^N Ac\|_{q,w},
\end{aligned} \tag{4.14}$$

and

$$\sum_{k\in N\mathbb{Z}^d}\Big(\sup_{|n|_\infty\ge|k|_\infty}|W_N(n)|\Big)<\infty. \tag{4.15}$$

Given any $i\in\mathbb{Z}^d$, let $n(i)$ be the unique integer in $N\mathbb{Z}^d$ with $i\in n(i)+\{0,\dots,N-1\}^d$. Then

$$\alpha_{n(i)}\le\sum_{|i'-i|_\infty<3N}w(i')\le(6N)^{dq}A_q(w)w(i)$$

by (4.1). This together with (4.14) implies that for any $c\in\ell^\infty$,

$$\begin{aligned}|c(i)| &\le (6N)^d(A_q(w))^{1/q}\alpha_{n(i)}^{-1/q}\|\Psi^N_{n(i)}c\|_{q,w}\\ &\le (6N)^d(A_q(w))^{1/q}\sum_{n'\in N\mathbb{Z}^d}W_N(n(i)-n')\\ &\quad\times\Big(\sum_{j\in\mathbb{Z}^d}h\big((j-n')/N\big)|(Ac)(j)|\Big)\\ &\le (6N)^d(A_q(w))^{1/q}\\ &\quad\Big\{\sum_{j\in\mathbb{Z}^d}\Big(\sum_{\epsilon\in\{-4,\cdots,4\}^d}W_N(n(i-j)+\epsilon N)\Big)|(Ac)(j)|\Big\}\\ &=: \sum_{j\in\mathbb{Z}^d}g(i-j)|(Ac)(j)|.\end{aligned} \tag{4.16}$$

Then the sequence $\{g(i)\}_{i\in\mathbb{Z}^d}$ just defined satisfies (4.2) and (4.3), the requirements in Lemma 4.5, by (4.15) and (4.16). □

Now we proceed to prove Theorem 4.1.

Proof of Theorem 4.1. By Theorem 3.2, it suffices to prove the conclusion for any infinite matrix $A\in\mathcal{B}$.

By (3.16),

$$\|Ac\|_{q',w'}\le 2^{2d}3^{d/q'}(A_{q'}(w'))^{1/q'}\|A\|_{\mathcal{B}}\|c\|_{q',w'}\quad\text{for all } c\in\ell^{q'}_{w'}. \tag{4.17}$$

Let $\{g(i)\}_{i\in\mathbb{Z}^d}$ be the sequence in Lemma 4.5, and set

$$A_0:=\sum_{k\in\mathbb{Z}^d}\Big(\sup_{|i|_\infty\ge|k|_\infty}g(i)\Big)<\infty.$$

Then

$$\begin{aligned}\|c\|_{q',w'} &\le \Big\|\Big(\sum_{j\in\mathbb{Z}^d}g(i-j)|(Ac)(j)|\Big)_{i\in\mathbb{Z}^d}\Big\|_{q',w'}\\ &\le 2^{2d}3^{d/q'}A_0(A_{q'}(w'))^{1/q'}\|Ac\|_{q',w'}\quad\text{for all } c\in\ell^\infty\cap\ell^{q'}_{w'},\end{aligned} \tag{4.18}$$

where the first inequality follows from (4.3) and the second inequality holds by (3.16). Combining (4.17) and (4.18) proves the $\ell^{q'}_{w'}$-stability for the infinite matrix $A \in \mathcal{B}$. □

Finally we prove Corollary 1.3.

Proof of Corollary 1.3. The necessity is well known, while the sufficiency follows from Theorem 3.2 and Corollary 1.7, whose proof will be given in the next section. □

5. Inverse-closedness

In this section, we prove Theorem 1.4, Corollaries 1.6 and 1.7, and the following Wiener's lemma for the subalgebra $\mathcal{B}_{p,u}$ of $\mathcal{B}(\ell^q_w)$.

Theorem 5.1. *Let $1 \le p, q < \infty$, w be a discrete A_q-weight, $u := (u(i,j))_{i,j\in\mathbb{Z}^d}$ be a p-submultiplicative weight matrix that satisfies* (3.1), (3.2), (3.3), (3.9) *and*

$$u(i,j) = u(j,i) \quad \text{for all } i, j \in \mathbb{Z}^d,$$

and let $v := (v(i,j))_{i,j\in\mathbb{Z}^d}$ be a companion weight matrix of the p-submultiplicative weight matrix u that satisfies (3.4). *If there exist $D \in (0,\infty)$ and $\theta \in (0,1)$ such that*

$$\inf_{N\ge 1} \big(A_N + B_N(p)t\big) \le D t^\theta \quad \text{for all } t \ge 1 \tag{5.1}$$

where

$$A_N := \sum_{|k|_\infty \le N} \sup_{|k|_\infty \le |i'-j'|_\infty \le N} v(i',j') \tag{5.2}$$

and

$$B_N(p) := \Big\| \Big(\sup_{|i'-j'|_\infty \ge |k|_\infty} v(i',j')(u(i',j'))^{-1} \Big)_{|k|_\infty \ge N/2} \Big\|_{p/(p-1)}, \tag{5.3}$$

then $\mathcal{B}_{p,u}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^q_w)$.

One may verify that the weight matrices $((1+|i-j|)^\alpha)_{i,j\in\mathbb{Z}^d}$ with $\alpha > d(1-1/p)$, and $(\exp(|i-j|^\delta))_{i,j\in\mathbb{Z}^d}$ with $\delta \in (0,1)$, and their companion weight matrices satisfy the conditions on weight matrices required in Theorem 5.1 [44]. Hence we have the following corollary of Theorem 5.1.

Corollary 5.2. *Let $1 \le p, q < \infty$, w be a discrete A_q-weight, and let u be either $((1+|i-j|_\infty)^\alpha)_{i,j\in\mathbb{Z}^d}$ with $\alpha > d(1-1/p)$ or $(\exp(|i-j|^\delta_\infty))_{i,j\in\mathbb{Z}^d}$ with $\delta \in (0,1)$. Then $\mathcal{B}_{p,u}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^q_w)$.*

5.1. **Proof of Theorem 1.4.** Let $A \in \mathcal{B}$ have an inverse $A^{-1} \in \mathcal{B}(\ell^q_w)$. Then $\|c\|_{q,w} \le \|A^{-1}\|_{\mathcal{B}(\ell^q_w)}\|Ac\|_{q,w}$ for all $c \in \ell^q_w$, where $\|\cdot\|_{\mathcal{B}(\ell^q_w)}$ is the operator norm on $\mathcal{B}(\ell^q_w)$. Therefore A has ℓ^q_w-stability. By Lemma 4.5, there exists a sequence $\{g(i)\}_{i\in\mathbb{Z}^d}$ such that (4.2) and (4.3) hold.

Write $A^{-1} := (b(i,j))_{i,j\in\mathbb{Z}^d}$, set $c_j := (b(i,j))_{i\in\mathbb{Z}^d}$, and for $l_0 \ge 1$ define $c_j^{l_0} := (b_{l_0}(i,j))_{i\in\mathbb{Z}^d}, j \in \mathbb{Z}^d$, where $b_{l_0}(i,j) := b(i,j)$ if $|i-j|_\infty \le l_0$ and 0 otherwise. Then $c_j^{l_0} \in \ell^\infty \cap \ell^q_w$ and

$$\lim_{l_0\to+\infty} \|c_j^{l_0} - c_j\|_{q,w} = 0. \tag{5.4}$$

Applying (4.3) to $c_j^{l_0}$ gives

$$|b_{l_0}(i,j)| \le \sum_{i'\in\mathbb{Z}^d} g(i-i')|(Ac_j^{l_0})(i')|, \qquad i \in \mathbb{Z}^d. \tag{5.5}$$

By (4.2), (5.4), and Theorem 3.2,

$$\begin{aligned}
& \sum_{i'\in\mathbb{Z}^d} g(i-i')|(A(c_j^{l_0}-c_j))(i')| \\
\le\ & w(i)^{-1/q}\Big\|\Big(\sum_{i''\in\mathbb{Z}^d} g(i'-i'')|(A(c_j^{l_0}-c_j))(i'')|\Big)_{i'\in\mathbb{Z}^d}\Big\|_{q,w} \\
\le\ & 2^{4d}3^{2d/q}w(i)^{-1/q}(A_q(w))^{2/q}\|A\|_{\mathcal{B}} \\
& \quad \times\Big\|\Big(\sup_{|j'|_\infty\ge|k|_\infty}|g(j')|\Big)_{k\in\mathbb{Z}^d}\Big\|_1\|c_j^{l_0}-c_j\|_{q,w} \\
\to\ & 0 \quad \text{as } l_0 \to +\infty.
\end{aligned} \tag{5.6}$$

Letting $l_0 \to +\infty$ in (5.5) and applying (5.6) gives

$$|b(i,j)| \le g(i-j) \quad \text{for all } i,j \in \mathbb{Z}^d. \tag{5.7}$$

Hence the conclusion $A^{-1} \in \mathcal{B}$ follows from (4.2) and (5.7). □

5.2. **Proof of Corollary 1.6.** Write $f(\xi) = \sum_{n\in\mathbb{Z}} a(n)e^{-\sqrt{-1}n\xi}$. Then $A := (a(i-j))_{i,j\in\mathbb{Z}}$ belongs to $\mathcal{B}$ and has bounded inverse in $\mathcal{B}(\ell^2)$. Moreover, $A^{-1} = (b(i-j))_{i,j\in\mathbb{Z}}$ for the sequence $b := (b(n))_{n\in\mathbb{Z}}$ determined by $1/f(\xi) = \sum_{n\in\mathbb{Z}} b(n)e^{-\sqrt{-1}n\xi}$. By Theorem 1.4, $A^{-1} \in \mathcal{B}$ which in turn proves the desired conclusion that $1/f \in A^*(\mathbb{T})$. □

5.3. **Proof of Corollary 1.7.** The necessity follows from Theorem 3.2. Now the sufficiency: Let $1 \le q < \infty$, w be a discrete A_q-weight, and let $A \in \mathcal{B}$ have ℓ^q_w-stability. Then A has ℓ^2-stability by Theorem 1.2, i.e., there exists a positive constant C such that

$$C^{-1}\|c\|_2 \le \|Ac\| \le C\|c\|_2 \quad \text{for all } c \in \ell^2.$$

This implies that A^*A has bounded inverse in $\mathcal{B}(\ell^2)$. On the other hand, A^*A belong to $\mathcal{B}$ by Proposition 2.4 and Theorem 3.2. Therefore

$$(A^*A)^{-1} \in \mathcal{B} \tag{5.8}$$

by Theorem 1.4. Now we prove that $B := (A^*A)^{-1}A^*$ is the desired left inverse of the infinite matrix A in $\mathcal{B}$. The conclusion that $B \in \mathcal{B}$ follows from (5.8), Proposition 2.4 and Theorem 3.2. From the definition of the infinite matrix B, it defines a left inverse in $\mathcal{B}(\ell^2)$, it belongs to $\mathcal{B}(\ell^q_w)$ by Theorem 3.2 and $B \in \mathcal{B}$, and the set $\ell^2 \cap \ell^q_w$ is dense in ℓ^q_w. Therefore the infinite matrix B is a left inverse in $\mathcal{B}(\ell^q_w)$. □

5.4. **Proof of Theorem 5.1.** To prove Theorem 5.1, we need a technical lemma. Similar results are established in [42, 44] for infinite matrices in the Gröchenig-Schur class $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ and the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, see also Remark 3.4.

Lemma 5.3. *Let* $1 \le p \le \infty$. *If the weight matrix* u *satisfies* (3.1), (3.2), (3.3), (3.9) *and* (5.1) *for some positive constants* $D \in (0, \infty)$ *and* $\theta \in (0, 1)$, *then*

$$\|A^2\|_{\mathcal{B}_{p,u}} \le 2^{2+2/p} 5^{(d-1)/p} D \|A\|^{1+\theta}_{\mathcal{B}_{p,u}} \|A\|^{1-\theta}_{\mathcal{B}(\ell^2)} \quad \text{for all } A \in \mathcal{B}_{p,u}. \tag{5.9}$$

Proof. Let $A := (a(i,j))_{i,j\in\mathbb{Z}^d} \in \mathcal{B}_{p,u}$, and let A_N and $B_N(p)$ be as in (5.2) and (5.3) respectively. Recall that $|a(i,j)| \le \|A\|_{\mathcal{B}(\ell^2)}$ for all $i, j \in \mathbb{Z}^d$. Then for $1 < p < \infty$,

$$\begin{aligned}
& \sum_{k'\in\mathbb{Z}^d} |a(i,k')|u(i,k')|a(k',j)|v(k',j) \\
\le\ & \inf_{N\ge 1} \Big\{ \|A\|_{\mathcal{B}(\ell^2)} \sum_{|k'-j|_\infty \le N} |a(i,k')|u(i,k')v(k',j) \\
& + \sum_{|k'-j|_\infty > N} |a(i,k')|u(i,k')|a(k',j)|v(k',j) \Big\} \\
\le\ & \inf_{N\ge 1} \Big\{ \|A\|_{\mathcal{B}(\ell^2)} \Big(\sum_{|k''-j|_\infty \le N} v(k'',j) \Big)^{(p-1)/p} \\
& \quad \times \Big(\sum_{|k'-j|_\infty \le N} (|a(i,k')|u(i,k'))^p v(k',j) \Big)^{1/p} \\
& \quad + \Big(\sum_{|k''-j|_\infty > N} |a(k'',j)|v(k'',j) \Big)^{(p-1)/p} \\
& \quad \times \Big(\sum_{|k'-j|_\infty > N} (|a(i,k')|u(i,k'))^p |a(k',j)|v(k',j) \Big)^{1/p} \Big\}.
\end{aligned}$$

Therefore we obtain

$$
\begin{aligned}
& \Big\{\sum_{k\in\mathbb{Z}^d}\sup_{|i-j|_\infty\ge |k|_\infty}\Big(\sum_{k'\in\mathbb{Z}^d}|a(i,k')|u(i,k')|a(k',j)|v(k',j)\Big)^p\Big\}^{1/p}\\
\le\ & \inf_{N\ge 1}\Big\{\|A\|_{\mathcal{B}(\ell^2)}(A_N)^{(p-1)/p}\\
&\qquad\times\Big(A_N\sum_{k\in\mathbb{Z}^d}\sup_{|i'-j'|_\infty\ge |k|_\infty/2}(|a(i',j')|u(i',j'))^p\\
&\qquad\quad+\|A\|_{\mathcal{B}_{p,u}}^p\sum_{k\in\mathbb{Z}^d}\sup_{|k|_\infty/2\le|i'-j'|_\infty\le N}v(i',j')\Big)^{1/p}\\
&\quad+\|A\|_{\mathcal{B}_{p,u}}^{(p-1)/p}(B_N(p))^{(p-1)/p}\\
&\qquad\times\Big(\|A\|_{\mathcal{B}_{p,u}}B_N(p)\sum_{k\in\mathbb{Z}^d}\sup_{|i'-j'|_\infty\ge|k|_\infty/2}(|a(i',j')|u(i,j'))^p\\
&\qquad\quad+\|A\|_{\mathcal{B}_{p,u}}^{p+1}\Big(\sum_{k\in\mathbb{Z}^d}\sup_{|i'-j'|_\infty\ge\max(|k|_\infty/2,N)}\\
&\qquad\qquad\Big(\frac{v(i',j')}{u(i',j')}\Big)^{p/(p-1)}\Big)^{(p-1)/p}\Big)^{1/p}\Big\}\\
\le\ & 2^{1+2/p}5^{(d-1)/p}\|A\|_{\mathcal{B}_{p,u}}\inf_{N\ge1}\Big(\|A\|_{\mathcal{B}(\ell^2)}A_N+B_N(p)\|A\|_{\mathcal{B}_{p,u}}\Big)\\
\le\ & 2^{1+2/p}5^{(d-1)/p}D\|A\|_{\mathcal{B}_{p,u}}^{1+\theta}\|A\|_{\mathcal{B}(\ell^2)}^{1-\theta}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \Big\{\sum_{k\in\mathbb{Z}^d}\sup_{|i-j|_\infty\ge |k|_\infty}\Big(\sum_{k'\in\mathbb{Z}^d}|a(i,k')|v(i,k')|a(k',j)|u(k',j)\Big)^p\Big\}^{1/p}\\
\le\ & 2^{1+2/p}5^{(d-1)/p}D\|A\|_{\mathcal{B}_{p,u}}^{1+\theta}\|A\|_{\mathcal{B}(\ell^2)}^{1-\theta}.
\end{aligned}
$$

Combining the above two estimates and applying (3.13) with $B=A$, we then get the desire conclusion (5.9) for $1<p<\infty$.

The conclusion (5.9) for $p=1$ and for $p=\infty$ can be established similarly. We omit the details here. □

Having the above technical lemma, we can combine the arguments in [42, 44] and Wiener's lemma for $\mathcal{B}$ to prove Theorem 5.1.

Proof of Theorem 5.1. Let $A\in\mathcal{B}_{p,u}$ and $A^{-1}\in\mathcal{B}(\ell^p_w)$. Then $A^{-1}\in\mathcal{B}\subset\mathcal{B}(\ell^2)$ by Theorems 1.4 and 3.2. This implies that $C_1I\le A^*A\le C_2I$ for some positive constants C_1 and C_2, where A^* is the conjugate transpose of the matrix A and I is the identity matrix. Now set

$$B := I - \frac{2}{C_1 + C_2} A^* A.$$

Then

$$\|B\|_{\mathcal{B}(\ell^2)} \le \frac{C_2 - C_1}{C_2 + C_1} := r_0 < 1. \tag{5.10}$$

On the other hand, $A^*A \in \mathcal{B}_{p,u}$ by Proposition 2.4 and Theorem 3.2, and $I \in \mathcal{B}_{p,u}$ by (3.9). This shows that

$$\|B\|_{\mathcal{B}_{p,u}} < \infty. \tag{5.11}$$

Given any integer $n \ge 1$, write $n = \sum_{l=0}^{l_0} \epsilon_l 2^l$ with $\epsilon_l \in \{0, 1\}$. Applying Theorem 3.2 and Lemma 5.3 iteratively gives

$$\begin{aligned}
\|B^n\|_{\mathcal{B}_{p,u}} &\le (C\|B\|_{\mathcal{B}_{p,u}})^{\epsilon_0} \|B^{n-\epsilon_0}\|_{\mathcal{B}_{p,u}} \\
&\le C(C\|B\|_{\mathcal{B}_{p,u}})^{\epsilon_0} (\|B\|_{\mathcal{B}(\ell^2)})^{(1-\theta)\sum_{l=0}^{l_0-1} \epsilon_{l+1} 2^l} \\
&\qquad \times (\|B^{\sum_{l=0}^{l_0-1} \epsilon_{l+1} 2^l}\|_{\mathcal{B}_{p,u}})^{(1+\theta)} \\
&\le \cdots \le C^{l_0} (C\|B\|_{\mathcal{B}_{p,u}})^{\sum_{l=0}^{l_0} \epsilon_l (1+\theta)^l} (\|B\|_{\mathcal{B}(\ell^2)})^{\sum_{l=0}^{l_0} \epsilon_l (2^l - (1+\theta)^l)} \\
&\le C^{\log_2 n} \big(C r_0^{-1} \|B\|_{\mathcal{B}_{p,u}}\big)^{n^{\log_2(1+\theta)}} r_0^n,
\end{aligned}$$

where $C = \max(2^{2+2/p} 5^{(d-1)/p} D, 2^{1+2/p} 5^{(d-1)/p} M_p(u))$. This together with (5.10) and (5.11) shows that

$$\begin{aligned}
\|A^{-1}\|_{\mathcal{B}_{p,u}} &= \|(A^*A)^{-1} A^*\|_{\mathcal{B}_{p,u}} \\
&= \frac{C_1 + C_2}{2} \Big\| A^* + \Big(\sum_{n=1}^{\infty} B^n\Big) A^* \Big\|_{\mathcal{B}_{p,u}} \\
&\le \frac{C_1 + C_2}{2} \Big\{ \|A^*\|_{\mathcal{B}_{p,u}} + C\|A^*\|_{\mathcal{B}_{p,u}} \\
&\qquad \times \Big(\sum_{n=1}^{\infty} C^{\log_2 n} \big(C r_0^{-1} \|B\|_{\mathcal{B}_{p,u}}\big)^{n^{\log_2(1+\theta)}} r_0^n \Big) \Big\} < \infty.
\end{aligned}$$

Hence the conclusion $A^{-1} \in \mathcal{B}_{p,u}$ is proved. □

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Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

E-mail address: qsun@mail.ucf.edu